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EXACT SOLUTIONS OF THE NON-LINEAR WAVE EQUATIONS ARISING IN MECHANICS*

N.A. KUDRYASHOV

Solitary and cnoidal waves solving three equations commonly used to describe wave processes in mechanics - the Burgers-Korteweg-de Vries, Kuramoto-Sivashinsky and Kawahara equations - are obtained by analytical means.

Weiss, Tabor and Carnevale proposed an effective method of determining the Bäcklund transformation and Lax pair for non-linear partial differential equations which are integrable by the inverse scattering method (ISM). The essence of the method is to expand the solution of the original equation in the neighbourhood of the singular manifold. For example, in the case of the Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=0 \tag{0.1}$$

which describes non-linear waves in a dispersive medium, one represents the solution in the form

$$u=u_0/F^2+u_1/F+u_2$$

substitutes this into (0.1) and then equates the coefficients of like powers of $F(x, t)$ to zero; this yields the Bäcklund transformation for the solutions of the KdV equation. The result is a much simpler approach to the solution of this equation.

It will be shown below that a similar method can be used to determine exact solutions for several non-linear equations which occur in mechanics and physics and are not integrable by the ISM.

1. ***Burgers-Korteweg-de Vries (BKdV) equation.*** This equation generalizes the KdV equation to take into account dissipative processes in wave propagation in shallow water /2/, in a liquid with gas bubbles /3/, in plasma /4/, etc. It differs from Eq.(0.1) in the additional term $\nu\partial^2u/\partial x^2$ on the right. It has been shown /5, 6/ that the solution of this equation can be expressed in the form

$$u(x, t)=12\beta\partial^2\ln F/\partial x^2-12\nu/5\partial\ln F/\partial x+u_3 \tag{1.1}$$

Using (1.1), we find exact solutions of the BKdV equation. We shall look for a solution in the travelling-wave coordinate system $u(x, t)=U(\xi)$, $\xi=x-c_0t$ (where c_0 is the velocity of the wave). The BKdV equation in this case may be written in the form

$$\beta U_{\xi\xi}''-\nu U_\xi'+{}^1/_2U^2-c_0U+q=0,\ q=\text{const} \tag{1.2}$$

Eq.(1.2), which describes a non-linear oscillator with damping due to friction forces, is a popular model in oscillation theory. Substituting into it the transform of the solution (1.1), i.e.,

$$U=C_1+12\beta\theta^2R(\theta),\quad R(\theta)=\frac{d^2\ln F}{d\theta^2},\quad \theta=\frac{5\beta}{\nu}\exp\left(\frac{\nu\xi}{5\beta}\right)$$

we get

$$\frac{\nu^2}{25\beta^2}R''+6R^2+\frac{1}{\beta}\left(C_1-c_0-\frac{6\nu^2}{25\beta}\right)R\theta^{-2}+\frac{1}{12\beta}\left(q-c_0c_1+\frac{C_1^2}{2}\right)\theta^{-4}=0 \tag{1.3}$$

This equation is convenient for approximate solution of the BKdV equation, but it can also be used to find exact solutions if q and C_1 are chosen that the expressions in parentheses in (1.3) vanish. In that case, multiplying both sides of Eq.(1.3) by R_θ' and integrating with respect to θ, we obtain the equation

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$$R'' + 100\beta^2\nu^{-2}(R^3 + C_2) = 0 \qquad (1.4)$$

The solution of the BKdV equation obtained through (1.4) can be expressed in terms of the Weierstrass function:

$$U(\xi) = C_1 - \frac{12\nu^2}{25\beta}\exp\left(\frac{2\nu\xi}{5\beta}\right)\wp\left(\frac{5\beta}{\nu}\exp\left(\frac{\nu\xi}{5\beta}\right), 0, C_3\right) \qquad (1.5)$$

If $C_3 = 0$ the solution of (1.5) takes the form of a wave front /6/:

$$u(x,t) = C_1 + \frac{12\nu}{5}\left(\frac{\nu}{5\beta} - k\right)\frac{E}{1+E} - \frac{12\nu^2E^2}{25\beta(1+E)^2} \qquad (1.6)$$

$$E = \exp(kx + \omega t), \quad k = \pm\frac{\nu}{5\beta}$$

$$\omega = -C_1k + \frac{6\nu^3}{125\beta^2}$$

2. ***The Kuramoto-Sivashinsky equation.*** Another formal generalization of the KdV equation is the Kuramoto-Sivashinsky equation, which describes non-linear waves in dissipative-dispersive media with instability: waves occurring in thin liquid films flowing down inclined planes /7, 8/, drift waves of an electrostatic potential in toroidal systems /9/, the concentration of material in chemical reactions /10, 11/, etc. The equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^4 u}{\partial x^4} = 0 \qquad (2.1)$$

It has been shown /5, 6/ that the solution of Eq.(2.1) can be expressed as

$$u(x,t) = \frac{15}{76}\left(16\alpha - \frac{\beta^2}{\gamma}\right)\frac{\partial \ln F}{\partial x} + 15\beta\frac{\partial^2 \ln F}{\partial x^2} + 60\gamma\frac{\partial^3 \ln F}{\partial x^3} + u_4 \qquad (2.2)$$

This transformation may be used to find analytical solutions. In particular, if $\alpha = \gamma = 1$, $\beta = 4$, $u_4 = C_1 = \text{const}$, substituting

$$u(x, t) = U(\xi), \ \xi = x - c_0t \qquad (2.3)$$

$$U(\xi) = C_1 + R + R_\xi', \ R = 60d^2 \ln F/d\xi^2$$

into Eq.(2.1), written in the travelling-wave coordinate system as

$$U_{\xi\xi\xi}''' + 4U_{\xi\xi}'' + U_\xi' + {}^1/_2U^2 - c_0U + q = 0 \qquad (2.4)$$

we obtain the equation

$$Z_{\xi\xi}'' + 5Z_\xi' + (5 - A)Z - {}^1/_5RZ + {}^3/_5\int ZR_\xi' d\xi = 0 \qquad (2.5)$$

Here

$$Z = R_{\xi\xi}'' + R^2/10 + AR - {}^5/_2(1 - A^2), \ A = (C_1 - c_0 + 1)/5 \qquad (2.6)$$

It follows from (2.5) and (2.6) that every solution of the equation $Z = 0$ is also a solution of Eq.(2.5), and hence the function $U(\xi)$ found from formula (2.3) is a solution of Eq.(2.1).

Multiplying (2.6) by R_ξ' and integrating the resulting expression with respect to ξ, we arrive at the equation

$$R_\xi'^2 + R^3/15 + AR^2 - 5(1 - A^2)R - 10D/3 = 0 \qquad (2.7)$$

The constant D in this equation is related to the constants q, c_0, C_1 and A by

$$D = C_1\left(c_0 - \frac{C_1}{2}\right) - \frac{5}{2}(5 - A)(1 - A^2) - q$$

The solution of Eq.(2.7) is expressed in terms of the Jacobian elliptic function:

$$R(\xi) = R_2 + (R_1 - R_2)\,\mathrm{cn}^2\left(\frac{\xi}{2}\sqrt{\frac{R_1 - R_3}{15}}, s\right), \quad s^2 = \frac{R_1 - R_2}{R_1 - R_3} \qquad (2.8)$$

where R_1, R_2 and R_3 $(R_1 \geqslant R_2 \geqslant R_3)$ are the real roots of the cubic equation

$$R^3 + 15AR^2 - 75(1 - A^2)R - 50D = 0$$

If $A = -1$ the solution (2.8) becomes a solitary wave:

$$R(\xi) = 15(1 - \mathrm{th}^2(\xi/2)) \tag{2.9}$$

Substituting (2.8) into (2.3), we find a solution of Eq.(2.1) in the form of a periodic (cnoidal) wave, which becomes a solitary wave if $A = -1$. These solutions are identical with the results of a numerical simulation of wave structures described by Eq.(2.1)*. (*Alekseyev A.A. and Kudryashov N.A., Numerical modelling of a selforganization process in dissipative-dispersive mdedia with instability. Preprint 027-88, Moskov. Inzhenerno-Fizicheskii Inst., Moscow, 1988.). Using transformations (2.2), one can also find other exact solutions of Eq. (2.1).

3. *The Kawahara equation.* Magneto-acoustic waves in plasma /12/, long waves in liquid under an ice cap /13/, coupled states of two solitons /14/, etc. are described by the non-linear equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = \frac{\partial^5 u}{\partial x^5} \tag{3.1}$$

This equation also has a Bäcklund-type transformation of the solutions:

$$u = \frac{280}{13}\frac{\partial^2 \ln F}{\partial x^2} - 280\frac{\partial^4 \ln F}{\partial x^4} + u_5$$

using which one can find a solution of the Kawahara Eq.(3.1) in the travelling-wave coordinate system. Substituting

$$u(x, t) = U(\xi),\ \xi = x - c_0 t \tag{3.2}$$

$$U(\xi) = C_1 + R/13 - R'',\ R = 280 d^2 \ln F/d\xi^2$$

into the equation

$$U_{\xi\xi\xi\xi}^{\mathrm{IV}} - U_{\xi\xi}'' - {}^1/_2 U^2 + c_0 U - q = 0$$

we see that it will have a solution (3.2) if

$$R_{\xi\xi}'' + 3R^2/140 - (A + 1/13)R - B = 0 \tag{3.3}$$

$$B = 5(c_0 - C_1) - \frac{35A}{3}\left(A + \frac{12}{13}\right) - \frac{180}{169},\quad A = \mathrm{const}$$

This implies the equation

$$R_\xi'^2 + R^3/70 - (A + 1/13)R^2 - 2BR - 28D/9 = 0 \tag{3.4}$$

where the constant D is related to A, c_0, C_1 and B by

$$D = 35A^2(1 - A) + 35A\left(C_1 - c_0 + \frac{12}{169}\right) - \frac{36B}{13} - 8AB$$

If R_1, R_2, and R_3 $(R_1 \geqslant R_2 \geqslant R_3)$ are the real roots of the cubic equation

$$R^3 - 70\left(A + \frac{1}{13}\right)R^2 - 140\left(BR + \frac{14D}{9}\right) = 0$$

then the solution of Eq.(3.4) is represented by formula (2.8) provided that

$$q = C_1\left(c_0 - \frac{C_1}{2}\right) + B(C_1 - c_0) - AB\left(A - \frac{12}{13}\right) + \frac{2D}{3}\left(A - \frac{9}{65}\right) - \frac{13B^2}{35}$$

If $A = B = D = C_1 = q = 0$, $c_0 = 36/169$, we deduce from (3.4) that

$$R(\xi) = \frac{70}{13}\mathrm{ch}^{-2}(\xi/\sqrt{52}) \tag{3.5}$$

Substituting (3.5) into (3.2), we find a solitary wave solution of the Kawahara equation:

$$U(\xi) = \frac{105}{169}\,\mathrm{ch}^{-4}(\xi/\sqrt{52})$$

Other values of the constants yield solutions of the Kawahara equation in the form of periodic waves.

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NON-AXISYMMETRIC BUCKLING OF SHALLOW SPHERICAL SHELLS*

I.M. BERMUS and L.S. SRUBSHCHIK

The buckling of elastic shallow orthotropic spherical shells subjected to a transverse load is investigated on the basis of geometrically non-linear equilibrium equations in a non-axisymmetric formulation. By using the method of finite differences and a continuation procedure in the prameters in combination with a Newton operator method an algorithm is constructed to determine the state of shell stress and strain in the pre- and post-critical stages.

The upper critical loads (CL) of spherical shells are determined for different external pressure distribution laws taking perturbing factors such as initial harmonic and azimuthal imperfection directions in the shape of the shell middle surface and analogous load deviations

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